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Planck's Constant, Torsion, and Space-Time Defects

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A possible way of building Planck's constant into the structure of space-time is considered. This is done by assuming that the torsional defect that intrinsic spin produces in the geometry is a multiple of the Planck length.

1. INTRODUCTION

Planck's constant $\hbar$ and the speed of light c play very fundamental roles in physics. They apply to the behavior of all matter through quantum mechanics and special relativity, respectively. Thus $\hbar$ and c logically precede other constants in physics, such as the coupling constants associated with the fundamental forces or dynamical quantities such as particle masses. Since $\hbar$ and c are so fundamental, one might expect them to be built into the local structure of space-time itself. Since the advent of special relativity (Einstein, 1905) and the conceptual revolution of taking four-dimensional space-time to be the arena of physics, we understand the geometrical role the speed of light plays. The speed of light is related to the topologically invariant signature and dimensionality of space-time. Planck's constant so far has not had a similar geometrical role. One might hope that the proper understanding of the geometrization of $\hbar$ might also have revolutionary consequences for the basic structure of the arena of physics. In particular, it might shed considerable light on quantum gravity.

The present paper presents one straightforward way in which $\hbar$ might be geometrized. It is necessarily somewhat heuristic and should be considered a first step only. No theory, heretofore, has even considered the geometrization of $\hbar$. String theory (Gross *et al.*, 1985*a*,*b*; Candelas *et al.*, 1985; Witten, 1985*a*,*b*), which is our deepest and most comprehensive theory

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of physics to date, quantizes the string in the same way as any other system. The geometrical origin of $\hbar$ itself is not addressed. If the existence of $\hbar$ is intimately related to the existence of spinors, which is likely not the case, then work on spinor structures in general relativity (Lichnerowicz, 1961, 1964; Milnor, 1962, 1963, 1965; Anderson *et al.*, 1966*a,b*; Penrose, 1968) could be related ultimately to the geometrical origin of $\hbar$.

Planck's constant is so ubiquitous in quantum mechanics that it is hard to see where $\hbar$ enters the theory at a fundamental level. $\hbar$ can enter at any one of several doors: the basic commutation relations, the path integral phase factor (Feynman and Gibbs, 1965), the de Broglie relation, and so on. I take the point of view in this paper that $\hbar$ comes into quantum mechanics through intrinsic spin and its interaction with the underlying geometry. This is reasonable since $\hbar$ is intimately related to spin and since the square of the spin commutes with all the dynamical variables that describe a particle and thus is a constant of the motion in all circumstances.

Once intrinsic spin is known to be quantized in units of $\hbar/2$ from geometrical arguments, one can argue that $\hbar$ must be present in the commutation relations for spin angular momentum. One can then argue that orbital angular momentum should obey the same commutation relations with $\hbar$ present. Since orbital angular momentum can be expressed as $\mathbf{r}\times\mathbf{p}$, this in turn says that $\hbar$ must be present in the fundamental commutation relations for x and P_x, for example. From there one is led to the $i\hbar\,\partial/\partial x$ operator description of P_x, to the de Broglie relation if one separately assumes that matter is described by waves, and to the Schrödinger equation. One clearly cannot get all of quantum mechanics—the interpretation of ψ, the way amplitudes are combined, and so on—but one does get $\hbar$ and at least some of the basic ideas that lead to quantum behavior.

Since we are interested in intrinsic spin and its interaction with the geometry, we are led consider a geometry with torsion present. As stressed by many authors, a geometry with torsion present is the natural extension of the Riemannian geometry of general relativity if sources with intrinsic spin are present. Early work on the relation of spin and torsion was done by Cartan (1922, 1923, 1924, 1925) and by Einstein (1955). Kibble (1961) and Sciama (1962, 1964) developed the gauge approach to a geometry with torsion present. They showed how natural torsion is if intrinsic spin is present. Trautman (1972, 1973*a–c*, 1975) analyzed and developed the theory in great detail using the techniques of modern differential geometry. I will use the work of Hehl (1973, 1974) and of Hehl *et al.* (1976) below.

I will go beyond the usual work on spin and torsion mentioned above and consider the possibility of a quantized defect in space-time in the following section, as a way of bringing $\hbar$ into space-time and hence into physics.

2. DEFECTS IN SPACE-TIME

Let us consider the possibility of a quantized defect in space-time. Kondo (1952), Bilby *et al.* (1955), and Kröner (1960) have considered the geometrical description of crystal dislocations or defects. In the limit of the dislocations having a continuous distribution, they found that torsion plays the role of the defect density. I will use the nice paper of Bilby *et al.* (1955) in a space-time rather than in a crystal context. They consider a small closed circuit and use Stokes' theorem to write

$$\mathscr{L}^\alpha = \int S_{\beta\gamma}{}^\alpha \, \mathrm{d}A^{\beta\gamma} \tag{1}$$

where

$$\mathrm{d}A^{\beta\gamma} \equiv \mathrm{d}x^\beta \wedge \mathrm{d}x^\gamma \tag{2}$$

is the element of area encircled by the loop and

$$S_{\beta\gamma}{}^\alpha \equiv \Gamma^\alpha_{[\beta\gamma]} \tag{3}$$

is the torsion associated with the connection $\Gamma^\alpha_{\beta\gamma}$. In the crystal case, $\mathscr{L}^\alpha$ represents the closure failure in the real dislocated crystal associated with a closed circuit in a perfect reference lattice. I will refer to this as the defect. The crystallographically equivalent steps are repeated in the real lattice as were used to obtain the closed circuit in the reference lattice. Closure does not happen, because of the defects in the real lattice.

For us the circuit is traversed using parallel displacement of vectors in space-time with the connection $\Gamma^\alpha_{\beta\gamma}$. The torsion is then the torsion present in space-time. Torsion has an intrinsic geometric meaning. From Misner *et al.* (1973), if torsion is absent $[U, V]$ and $\nabla_U V - \nabla_V U$ represent the same vector, where U and V are vectors. $\nabla_U V - \nabla_V U - [U, V] = 0$ represents a closed loop. If torsion is present (Wald, 1984), we get

$$U^a \nabla_a V^c - V^a \nabla_a U^c - [U, V]^c = S^c_{ab} U^a V^b \tag{4}$$

in terms of components, and the torsion term represents the failure of the loop to close in analogy with the crystal case.

I want to use (1) to quantize intrinsic spin. To do this, I want to relate the torsion $S_{\beta\gamma}{}^\alpha$ to intrinsic spin. I follow the work of Hehl *et al.* (1976) and use their notation and sign conventions. The signature is $(-1, +1, +1, +1)$. I first need to define a spin angular momentum tensor. To do this I write down a special relativistic Lagrangian density of matter $\mathscr{L}$ for a matter field ψ in Cartesian coordinates. This depends on the Minkowski metric. Gravity is then turned on by replacing the Minkowski metric with the space-time metric $g_{\alpha\beta}$ and by replacing $\partial\psi$ with the covariant derivative containing $\Gamma^{\gamma}_{\alpha\beta}$. The metric energy-momentum tensor is then

$$\sqrt{-g}\sigma^{\alpha\beta} \equiv 2\delta\mathscr{L}/\delta g_{\alpha\beta} \tag{5}$$

and the spin angular momentum tensor is

$$\sqrt{-g}\ \tau_{\alpha}{}^{\beta\gamma} \equiv \delta\mathscr{L}/\delta K_{\gamma\beta}{}^{\alpha} \tag{6}$$

where the contortion is

$$K_{\alpha\beta}{}^{\gamma} \equiv -S_{\alpha\beta}{}^{\gamma} + S_{\beta}{}^{\gamma}{}_{\alpha} - S^{\gamma}{}_{\alpha\beta} \tag{7}$$

I use the usual Hilbert action, written in terms of the above connection, for the gravitational field. Varying the total action, composed of matter plus gravitational parts with coupling constant k, with respect to $g_{\alpha\beta}$ and $\Gamma^{\gamma}_{\alpha\beta}$ gives the field equations

$$G^{\alpha\beta}(\{\cdot\}) = k\tilde{\sigma}^{\alpha\beta} \tag{8}$$

and

$$T^{\alpha\beta\gamma} = k\tau^{\alpha\beta\gamma} \tag{9}$$

where

$$T_{\alpha\beta}{}^{\gamma} \equiv S_{\alpha\beta}{}^{\gamma} + 2\delta^{\gamma}_{[\alpha}S_{\beta]\pi}{}^{\pi} \tag{10}$$

is the modified torsion. $\tilde{\sigma}^{\alpha\beta}$ is the effective energy-momentum tensor which sources the usual Einstein equation written in terms of $\{{}^{\alpha}_{\beta\gamma}\}$ Christoffel symbols. $\tilde{\sigma}^{\alpha\beta}$ is $\sigma^{\alpha\beta}$ from (5) plus terms depending on the torsion (Hehl *et al.*, 1976). I will take the coupling constant $k = 8\pi G/c^3$, so that $\tau^{\alpha\beta\gamma}$ has dimensions of angular momentum/unit volume. $S_{\alpha\beta}{}^{\gamma}$ has dimensions of length^{-1} as usual. Below, I am interested in (9), which gives the torsion in terms of the spin angular momentum tensor, since I am trying to relate torsion to intrinsic spin. Equation (9) contains no derivatives, so that the modified torsion is zero outside of the sources. Torsion does not propagate. Since spin is highly localized, this means that torsion is highly localized.

To go further, one needs to know more about the properties of the spin angular momentum tensor $\tau^{\alpha\beta\gamma}$. If one takes the matter distribution to be the "perfect fluid" of general relativity generalized to have nonvanishing spin, then

$$\tau_{\alpha\beta}{}^{\gamma} = \tau_{\alpha\beta}u^{\gamma} \tag{11}$$

with $\tau_{\alpha\beta} = -\tau_{\beta\alpha}$, where u^{γ} is the timelike four-velocity. In order to ensure that the equations of motion for the particles are integrable, one also must have (Frenkel, 1926)

$$\tau_{\alpha\beta}u^{\beta} = 0 \tag{12}$$

This means that $\tau_{\alpha\beta}{}^{\gamma}$ is completely antisymmetric (dual to an axial vector). If one considers a Dirac field, $\tau_{\alpha\beta}{}^{\gamma}$ is also totally antisymmetric (Hehl *et al.*, 1976). If $\tau_{\alpha\beta}{}^{\gamma}$ is totally antisymmetric, then so is $T^{\alpha\beta\gamma}$ from (9) and

$$T_{\alpha\beta}{}^{\gamma} = S_{\alpha\beta}{}^{\gamma} \tag{13}$$

from (10). Thus one has

$$S_{\alpha\beta}{}^{\gamma} = k\tau_{\alpha\beta}{}^{\gamma} \tag{14}$$

as the desired relationship between the spin angular momentum tensor and torsion. These are both totally antisymmetric and the γ index is timelike from (11). These are the properties needed below. I reiterate that $\tau_{\alpha\beta}{}^{\gamma}$ refers to intrinsic spin and not orbital angular momentum. This is clear from the whole foundation of the theory (Cartan, 1922, 1923, 1924, 1925).

If one now puts (14) together with (1), one has

$$\mathscr{L}^{\alpha} = k \int \tau_{\beta\gamma}{}^{\alpha}\, \mathrm{d}A^{\beta\gamma} \tag{15}$$

$\mathscr{L}^{\alpha}$ represents the closure failure or the "defect" upon going around a loop whose interior is $\mathrm{d}A^{\beta\gamma}$, produced by the spin angular momentum tensor $\tau_{\alpha\beta}{}^{\gamma}$. The quantity $\mathscr{L}^{\alpha}$ has dimensions of length and is timelike from (11). Equation (15) is essentially a classical expression, even though one is talking about intrinsic spin, which is not really a classical concept.

I now assume that $\tau_{\alpha\beta}{}^{\gamma}$ is produced by a single particle with intrinsic spin. I want to rewrite (15) in terms of the spin of the particle. It is easiest to see what is going on by going to the center-of-mass frame of the particle, where $u^0 = 1$ and $u^i = 0$. Since $\mathscr{L}^{\alpha}$ is timelike and totally antisymmetric, (15) becomes

$$\mathscr{L}^{0} = k \int \tau_{ij}{}^{0}\, \mathrm{d}x^i \wedge \mathrm{d}x^j \tag{16}$$

where $i, j = 1, 2, 3$ are spatial indices now. Multiply (16) by $\mathrm{d}x^l$, where $l \neq i$ and $l \neq j$, and integrate to get

$$\int \mathscr{L}^{0}\, \mathrm{d}x^l = k \int \tau_{ij}{}^{0}\, \mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^l \tag{17}$$

This can be written as

$$\int \mathscr{L}^{0}\, \mathrm{d}x^l = k \int \tau_{ij}{}^{0} \varepsilon^{0ijl}\, \mathrm{d}^3x \tag{18}$$

where d^3x is the usual three-dimensional spatial Reimann integral. Now the integral on the right-hand side of (18) is just twice the intrinsic spin s^l

of the particle in the center-of-mass frame. One can see this from Weinberg's (1972) definition of intrinsic spin,

$$s^\alpha = \tfrac{1}{2}\varepsilon^{\alpha\beta\gamma\delta} J_{\beta\gamma} u_\delta \tag{19}$$

where the present $\tau_{\alpha\beta}$ is the intrinsic spin analogue of the density of his $J_{\alpha\beta}$. Thus, one ends up with

$$\int \mathscr{L}^0 \, dx^l = 2ks^l \tag{20}$$

as the derived relation between the spin of the particle and the defect in the center-of-mass frame. The covariant expression is

$$\int \mathscr{L}^\alpha \, dx^\varepsilon = k\varepsilon^{\alpha\beta\gamma\varepsilon} s_{\beta\gamma} \tag{21}$$

where $\mathscr{L}^\alpha$ is timelike and

$$s^k \equiv \tfrac{1}{2}\varepsilon^{0ijk} s_{ij} \tag{22}$$

in the center-of-mass frame.

So far we have the interesting relation (20), but nothing radically new. I now make the key assumption that $\hbar$ enters physics through the integral defect on the left-hand side of (20). I assume that the timelike defect $\mathscr{L}^\alpha$ of space-time is quantized in units of the Planck length L_P. Thus, I assume that space-time cannot have just any defect, but only multiples of $L_P \equiv (\hbar G/c^3)^{1/2}$. I use L_P because it is the natural unit of length that can be built up from $\hbar$, G, c. It is the only distance scale available. Also, defects would be expected to be on this order because space-time is likely a topological foam on this distance scale (Misner *et al.*, 1973). This assumption then gives

$$\mathscr{L}^\alpha \mathscr{L}_\alpha = -n^2 L_P^2 \tag{23}$$

or

$$\mathscr{L}^0 = nL_P \tag{24}$$

in the center-of-mass frame of the particle. This assumption builds $\hbar$ fundamentally into the torsion/defect properties of space-time. This geometrizes $\hbar$ in somewhat the same way that Minkowski space geometrizes c. The present assumption is analogous to the fundamental assumptions that go into Minkowski space.

The test of assumptions at this fundamental level is whether or not they reproduce the physics that we observe. I show below, using the relation (2), that this assumption leads to the observed quantization of spin.

I also need to use the fact that torsion does not propagate and is nonzero only where the particle spin is nonzero. Thus, $\mathscr{L}^\alpha$ is nonzero only where spin is nonzero. Spin, however, for a particle like an electron is viewed as being pointlike in extent. Again using the idea that space-time is a topological form of distance scale $\sim L_P$, it is reasonable to assume that spin and hence $\mathscr{L}^\alpha$ are nonzero over this distance scale. This is clearly heuristic, but it is difficult to do better. The integral in (20) thus has an extent $\sim L_P$.

Let me now put this together. If the area $\mathrm{d}A^{\beta\gamma}$ is chosen to be in the 1, 2 plane in the center-of-mass frame of the particle, then (20) gives

$$\int \mathscr{L}^0 \, \mathrm{d}x^3 = 2ks^3 \tag{25}$$

The assumptions of the quantization of the timelike space-time defect (24) and on the extent of the spatial region over which $\mathscr{L}^\alpha$ is nonzero then give for (25)

$$(nL_P)L_P \sim ks^3 \tag{26}$$

Using the expressions for L_P and for k above then gives

$$s^3 \sim n\hbar \tag{27}$$

to within uncertain numerical factors of order 1.

Thus, the simple geometrical assumption (23) does lead to quantization of angular momentum. From the introduction, this leads naturally to $\hbar$ in the commutation relations, to quantum operators for momentum and position, and to the Schrödinger equation. We have geometrized $\hbar$ and have brought it into physics. The above is heuristic, but hopefully contains conceptual ideas that will lead eventually to a deeper geometrical understanding of Planck's constant. It is suggestive that $\hbar$ seems to be intimately related to geometrical structures of a size on the order of the Planck length. The fact that in this paper $\hbar$ is related to a quantized timelike vector with dimensions of length also suggests a possible relationship with the work on discrete time by Lee (1983).

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